1. Graph of $z = f(x, y)$
2. limit
3. continuous
4. linear approximation
5. differentiable
6. Chain Rule
7. Directional derivatives

Differential Calc

- def. Scalar function

从 graph 入手 $\begin{cases} \text{level curve 等高线 (横切)} & z \text{ 为 constant} \\ \text{cross section (竖切)} & x/y \text{ 为 constant} \end{cases}$

$$\boxed{\begin{array}{c} f(x,y) = z \\ \uparrow \qquad \uparrow \\ f: \text{domain} \to \text{range} \end{array}}$$

- 判断是否 cts.

满足：

1) $f(a,b)$ defined
2) $\lim\limits_{(a,b)}$ exist
3) $\lim\limits_{(a,b)} f(x,y) = f(a,b)$

▷ determine whether $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exist.

$f(x,y)$ cont at (a,b)?

Y → exisit (用 limit theorem, 直接代)

N → approach limit along $y=f(x)$ (可用 L'HR)

取 $y=f(x)$ 尽量消掉 x.

approach limit along $y=f(x)$: N → limit DNE

approach limit along $y=f(x)$: Y → Squeeze theorem

$||a|-|b|| \le |a \pm b| \le |a|+|b|$ ----▷ Squeeze theorem

Squeeze theorem: Y → exist

Squeeze theorem: N → 无法得出结论

- 判断是否 diff' at (a,b)

判断 $\lim\limits_{(x,y)\to(a,b)} \dfrac{R_{(a,b)}(x,y)}{\|(x,y)-(a,b)\|} \overset{?}{=} 0$

$L_{(a,b)}(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$R_{(a,b)}(x,y) = f(x,y) - L_{(a,b)}(x,y)$

→ continuity theo / squeeze theo $= 0$

→ approach along ... $\neq 0$

- theorem

① diff $\Rightarrow$ cont

② f_x & f_y cont at (a,b) $\Rightarrow$ $f(x,y)$ diff' at (a,b)

③ MVT: $f(t)$ cont on $[t_1, t_2]$. diff' on (t_1, t_2)

then $\exists\, t_0 \in (t_1, t_2)$ s.t. $f(t_2) - f(t_1) = f'(t_0)(t_2 - t_1)$

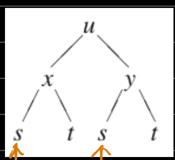
一 计算

求导

用于 i) 求 $f_x(a,b)$

ii) 证明 $f_x(a,b)$ exist

- partial derivatives: $f_x = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$
- Chain Rule $G'(t_0) = f_x(a,b)\,x'(t_0) + f_y(a,b)\,y'(t_0)$



$\frac{d}{dt} f(g(t)) = \nabla f(g(t)) \cdot \frac{dg}{dt}(t)$ $\quad g(t)=(x(t),y(t))$

eg. $\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}$

求近似

$f(x,y) \approx L_{(a,b)}(x,y)$

- Hessian matrix $Hf(x,y) = \begin{array}{c|c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \end{array}$ If $f_{xy}=f_{yx}$ Clairaut's theo)

→ linear approximation 近似:

- linearization $L_{(a,b)}(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$R_{1(a,b)}(x,y) = \frac{1}{2}[f_{xx}(x-a)^2 + 2f_{xy}(x-a)(y-b) + f_{yy}(y-b)^2]$

linear approx $f(\vec{x}) \approx L_{\vec{a}}(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a})(\vec{x}-\vec{a})$

→ Taylor Polynomial

- Taylor Polynomial $P_{2(a,b)}(x,y) = L_{(a,b)} + R_{1(a,b)}(x,y)$
- Taylor theorem 存在 (c,d) 在 $(a,b)\sim(x,y)$ 的线段上.

s.t $R_{1(a,b)}(x,y) = \frac{1}{2}[f_{xx}(c,d)(x-a)^2 + 2f_{xy}(c,d)(x-a)(y-b) + f_{yy}(c,d)(y-b)^2]$

- $\exists M>0$ s.t $|R_{1(a,b)}(x,y)| \le M\|(x,y)-(a,b)\|^2$

- k-th degree polynomial $P_{k,(a,b)}(x,y) = \sum_{|\alpha|\le k} \partial^\alpha f(a,b) \frac{[(x,y)-(a,b)]^\alpha}{\alpha!}$

$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$

$\partial^\alpha f = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots f$

$(\vec{x}-\vec{a})^\alpha = (x_1-a_1)^{\alpha_1}(x_2-a_2)^{\alpha_2}\cdots$

→ 结合图像

- tangent plane 切面 : $z = L_{(a,b)}$
- orthogonality theo : $f_{\text{level curve}} \perp g_{\text{level curve}} \Rightarrow \nabla f \cdot \nabla g = 0$
- directional derivative : $D_{\hat{u}} f(a,b) = \frac{d}{ds} f(a+su_1,\ b+su_2)\Big|_{s=0}$ (def)

$$= \nabla f(a,b) \cdot \hat{u} \qquad \hat{u}: \text{unit}$$

$(D_{\hat{u}} f(a,b) = \|\nabla f(a,b)\| \|\hat{u}\| \cos\theta)$

GRC : $D_{\hat{u}} f(a,b) = \|\nabla f(a,b)\|$

- critical point

定义 $f_x = 0$ / DNE 且 $f_y = 0$ / DNE

quadratic form $Q(u,v) = [u \ v] \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = a_{11}u^2 + 2a_{12}uv + a_{22}v^2$

$Q > 0$	pos defi	local min	$\det(Hf) > 0$	$a_{11} > 0$
$Q < 0$	neg defi	local max	$\det(Hf) > 0$	$a_{11} < 0$
$Q < 0 / > 0$	indef	saddle	$\det(Hf) < 0$	
其它	semidef	degenerate critical	$\det(Hf) = 0$	

local max/min $f_x(a,b) = f_y(a,b) = 0$ or $f_x(a,b) / f_y(a,b)$ DNE

saddle $f(x_1, y_1) > f(a,b)$ $f(x_2, y_2) < f(a,b)$

degenerate critical 2nd partial derivate test 无法得结论

要自己解出 (a,b) 的值

- convex function 凸形函数

convex	$f''(x) \geq 0$	$Hf(x,y)$ pos semi-def
strictly convex	$f''(x) > 0$	$Hf(x,y)$ pos def

→ ① $f(x,y) > L_{(a,b)}(x,y)$ $\forall (x,y) \neq (a,b)$

② $f(a_1 + t(b_1 - a_1),\ a_2 + t(b_2 - a_2)) < f(a_1, a_2) + t[f(b_1, b_2) - f(a_1, a_2)]$

一实际问题

EVT: f cont on I. $c_1, c_2 \in I$. $\Rightarrow f(c_1) \le f(x) \le f(c_2)$

求 extreme value in S: ① 找出 $\begin{cases} \text{critical point} \\ \text{boundary point} \end{cases}$

② 比较

Lagrange multiplier: $\boxed{\begin{array}{ll} \max/\min & f(x,y). \\ \text{s.t.} & g(x,y)=k. \end{array}}$ 求 max/min.

① 解 $\nabla f(x,y) = \lambda \nabla g(x,y)$ $\quad g(x,y)=k$

② 解 $\nabla g(x,y) = (0,0)$ $\quad g(x,y)=k$

③ check endpoints (若为 closed curve, 则无 endp.).

④ 比较

Integral Calc

— mapping

derivative matrix of $F(x,y)=(f(x,y),g(x,y))$: $DF=\begin{array}{c|cc} & x & y \\ \hline f & \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ g & \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{array}$

composite mapping : $D(F\circ G)(x,y)=DF(u,v)DG(x,y)$

invertible mapping : $(x,y)=F^{-1}(u,v)$

Jacobian $\frac{\partial(u,v)}{\partial(x,y)}=\det[DF(x,y)]=\det\left(\begin{array}{c|cc} & x & y \\ \hline u & \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ v & \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{array}\right)$

construct mapping 求boundary line，写作 $(u,v)=F(x,y)=\cdots$

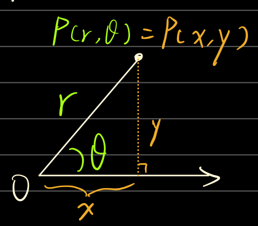
— double integral

absolute value inequality $\left|\iint_D f\,dA\right| \le \iint_D |f|\,dA$

$$\iint_D f(x,y)\,dA=\int_{x_l}^{x_u}\int_{y_l(x)}^{y_u(x)} f(x,y)\,dy\,dx$$

function 放里面

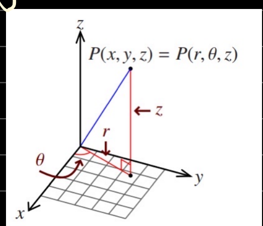
polar coordinate



$x=r\cos\theta$
$y=r\sin\theta$

$r=\sqrt{x^2+y^2}$

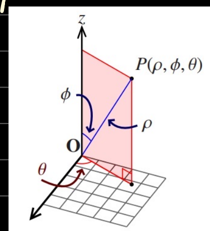
cylindrical coordinate



$x=r\cos\theta$
$y=r\sin\theta$
$z=z$

$r=\sqrt{x^2+y^2}$

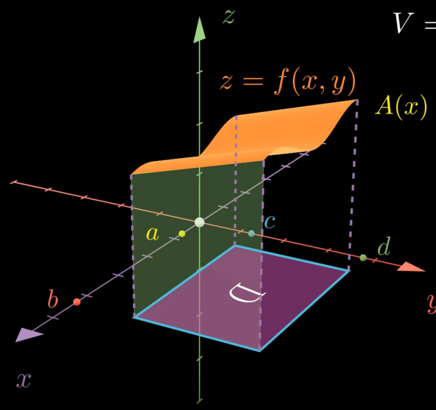
spherical coordinate



$x=\rho\sin\phi\cos\theta$
$y=\rho\sin\phi\sin\theta$
$z=\rho\cos\phi$

$\rho=\sqrt{x^2+y^2+z^2}$

- triple integral



$$V = \int_a^b A(x)dx$$

$$A(x) = \int_c^d f(x,y)dy$$

$$\iiint_D f(x,y,z)\,dV$$

- 当 approach limit though $y=mx$ 时. 注意有时 不存在

  eg. $\sqrt{mx}$. mx 存在条件是 ≥ 0. Week U2&U3/Q1

- Squeeze theorem 能用到 ↓

  triangle inequality $|a+b| \leq |a|+|b|$

  cosine inequality $2|x||y| \leq x^2+y^2$

  用于 squeeze theorem

- 遇到 "$|x-1|$" → 判断 $\lim_{x\to1^+}$ 和 $\lim_{x\to1^-}$

- 如果能 $f(x,y)$ 能化简得到 polynomial，则不用 squeeze theorem Quiz 1/Q2(a)

- 求 $\frac{\partial f}{\partial x}$ of $f(x,y)=\sqrt{|xy|}$

  先将 $|xy|$ 求导↓.

  $$\frac{\partial}{\partial x}(|xy|) = \frac{\partial}{\partial x}(|x||y|) = |y|\frac{x}{|x|}$$

  use chains rule ↓ (treat y as constant)

  $$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{|xy|}}|y|\cdot\frac{x}{|x|}$$

  $$= \frac{x\sqrt{|y|}}{2|x|\sqrt{|x|}}$$

- A silo consists of a circular cylinder of radius 5 meters, and height 25 meters, capped by a hemisphere. Suppose that the radius is decreased by 5 centimeters and the height of the cylinder is increased by 10 centimeters. Use the linear approximation to estimate the change in volume. Use $\pi = 3.14$ and round your answer to 1 decimal.

  $\Delta V \approx$

- 求 $f(x,y)=x|y-1|$ in f_y.

  $$f_y = x\frac{y-1}{|y-1|}$$

10章之前